

# Abstract Kleisli structures on 2-categories

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## Plan

1. Review & reformulate the definition of AKS on categories

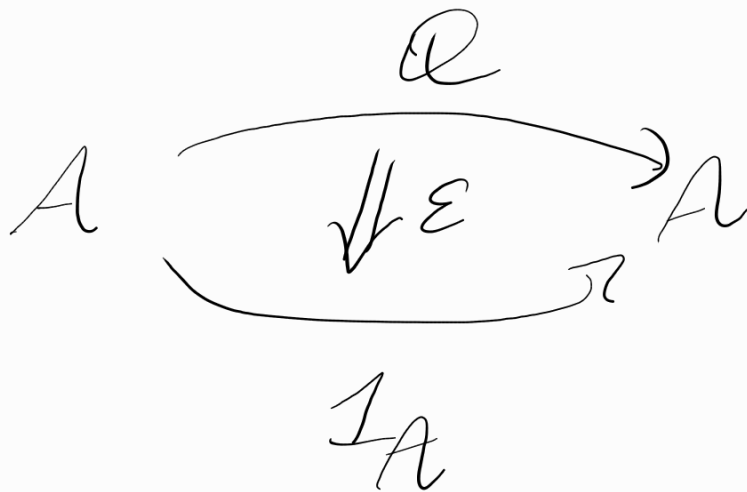
2. Review & extend theory of AKS on categories

3. Define AKS on 2-categories

4. State analogous results for AKS on 2-categories.

Def<sup>n</sup> (Führmann 1999)

An AKS consists of



"force"

$$X \xrightarrow{\phi_X} \mathcal{Q}X$$

"think"

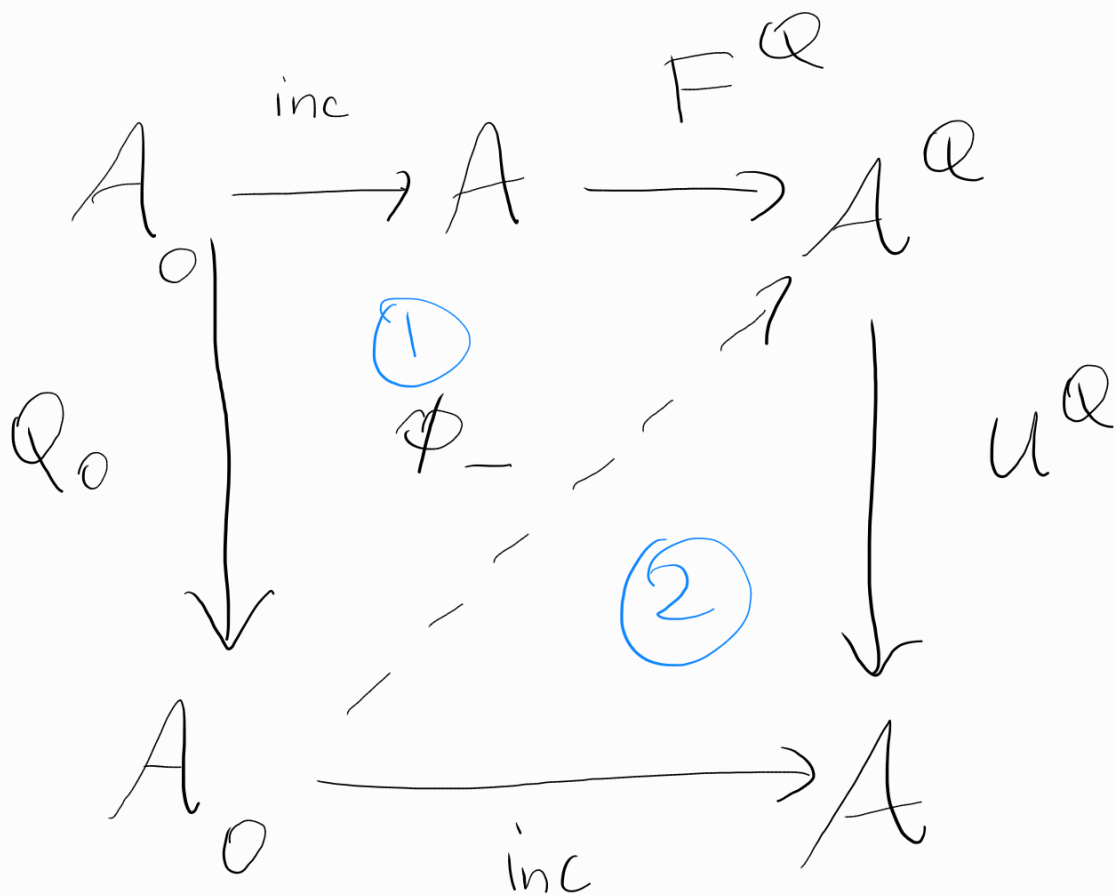
such that  $(A, \mathcal{Q}, \varepsilon, \phi_{\mathcal{Q}})$

is a comonad &

$\phi_X : X \rightarrow \mathcal{Q}X$  is a

coalgebra.

Part 1 Reformulating the definition of AKS on categories



①  $\phi_{QX}$  multiplication for a comonad.

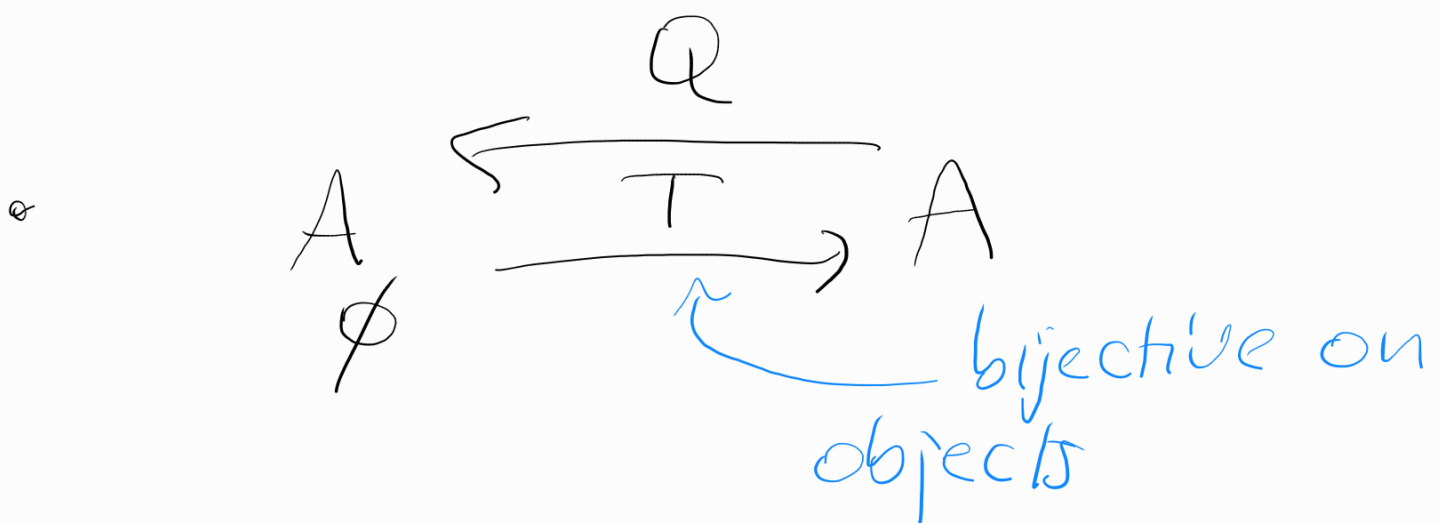
②  $\phi_X$  is a coalgebra.

- Note that  $\phi_X \circ \phi_X \circ X \rightarrow \mathbb{Q}X$   
not natural in  $X$

- morphisms  $f: X \rightarrow Y$  satisfying

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \phi_X \downarrow & & \downarrow \phi_Y \\
 \mathbb{Q}X & \xrightarrow{\mathbb{Q}f} & \mathbb{Q}Y
 \end{array}$$

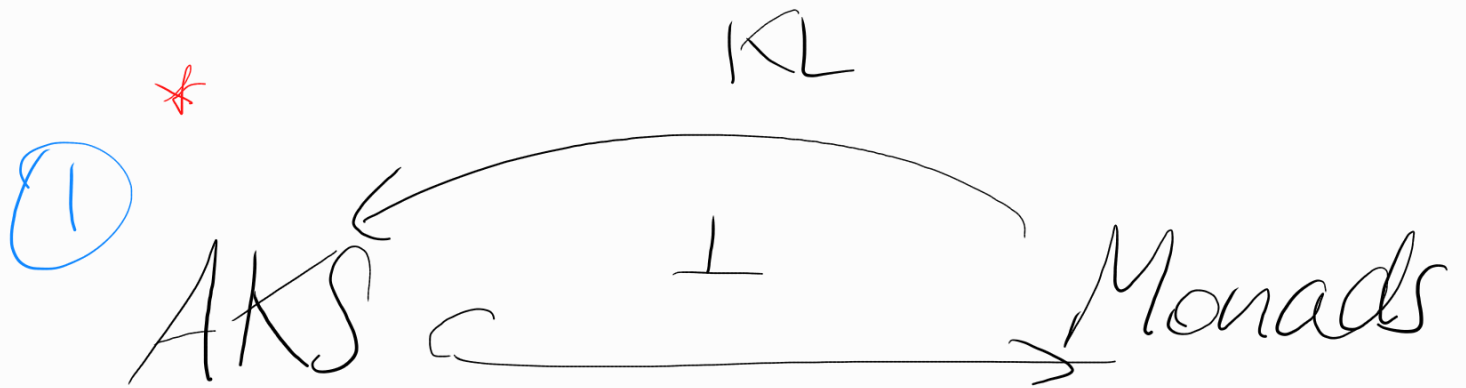
are called "thunkable."



- all examples are Kleisli categories of monads.



Part 2 Extending results on  
AKS on categories.



$$(A, Q, e, \delta, \phi) \longmapsto A_\phi$$

② The essential image is  
characterised by any of the  
following equivalent conditions

•  $X \xrightarrow{\eta_X} TX \xrightarrow[\cong]{T\eta_X} T^2X$  equalizer

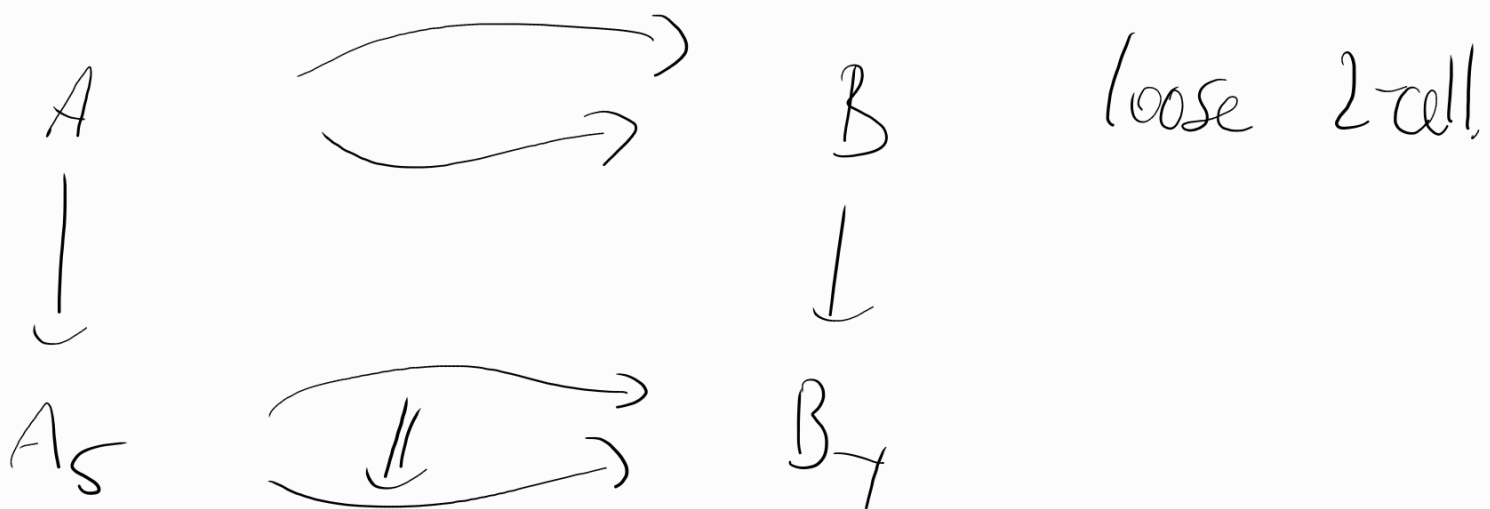
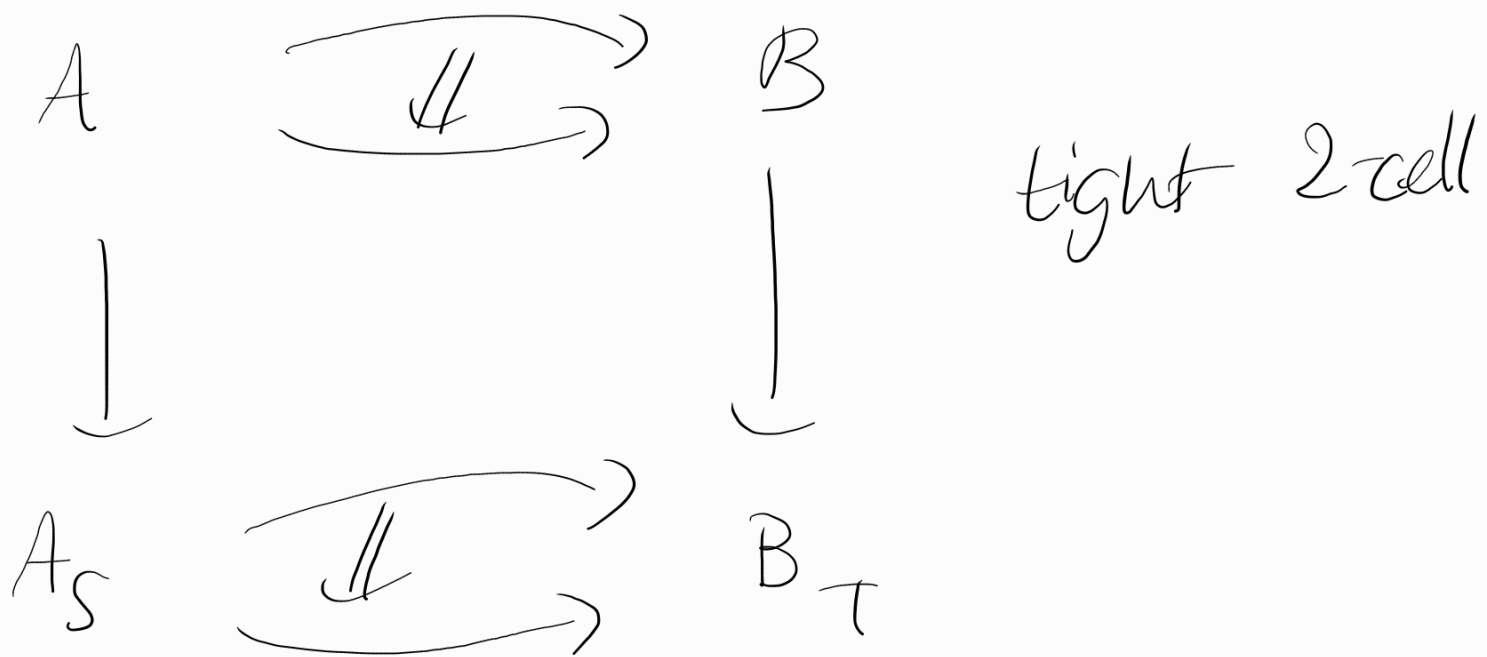
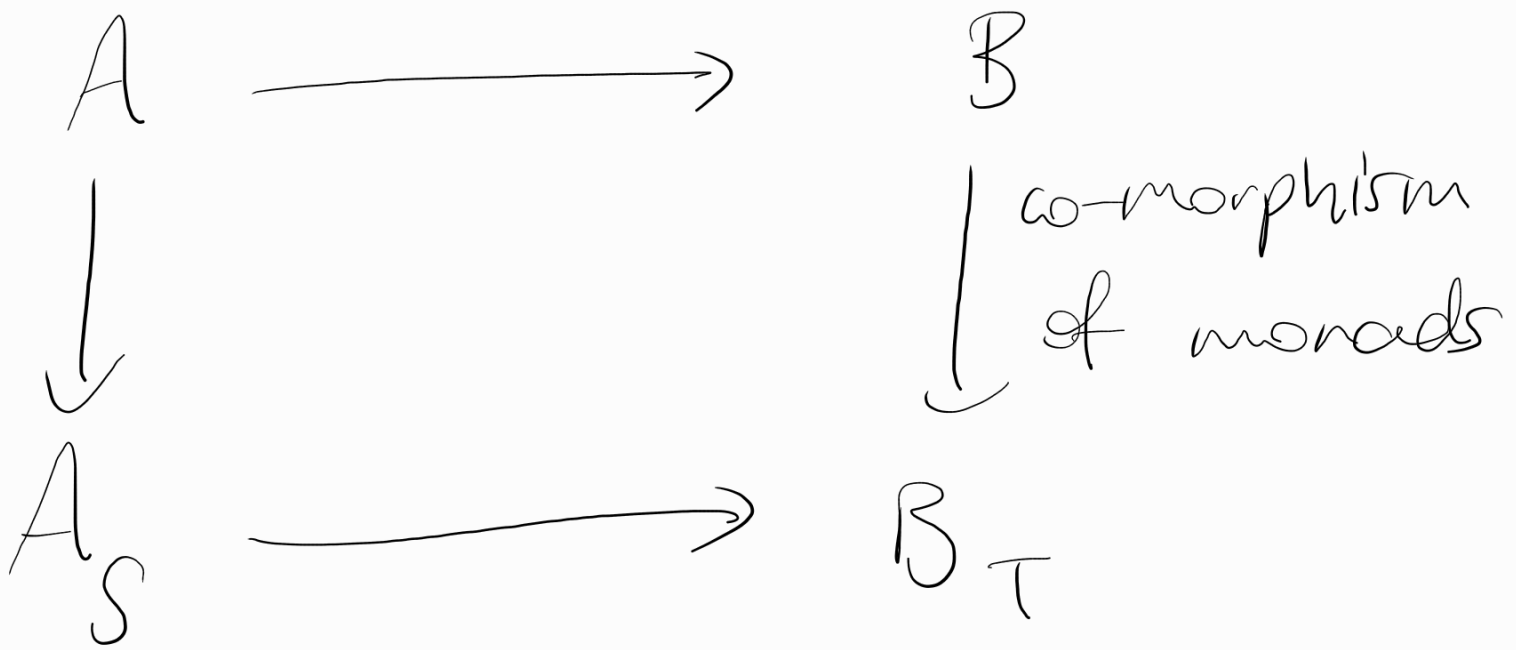
•  $B \rightarrow B_T$  faithful & full on thinkable morphisms

•  $B \rightarrow (B^T)^T$  fully faithful

•  $B \rightarrow (B_+)^T$  fully faithful

\* with strict morphisms of monads Fiehrmann (1999)

\* with co-morphisms of monads & either possible 2-cell (M. 2024) Prop 2.7.

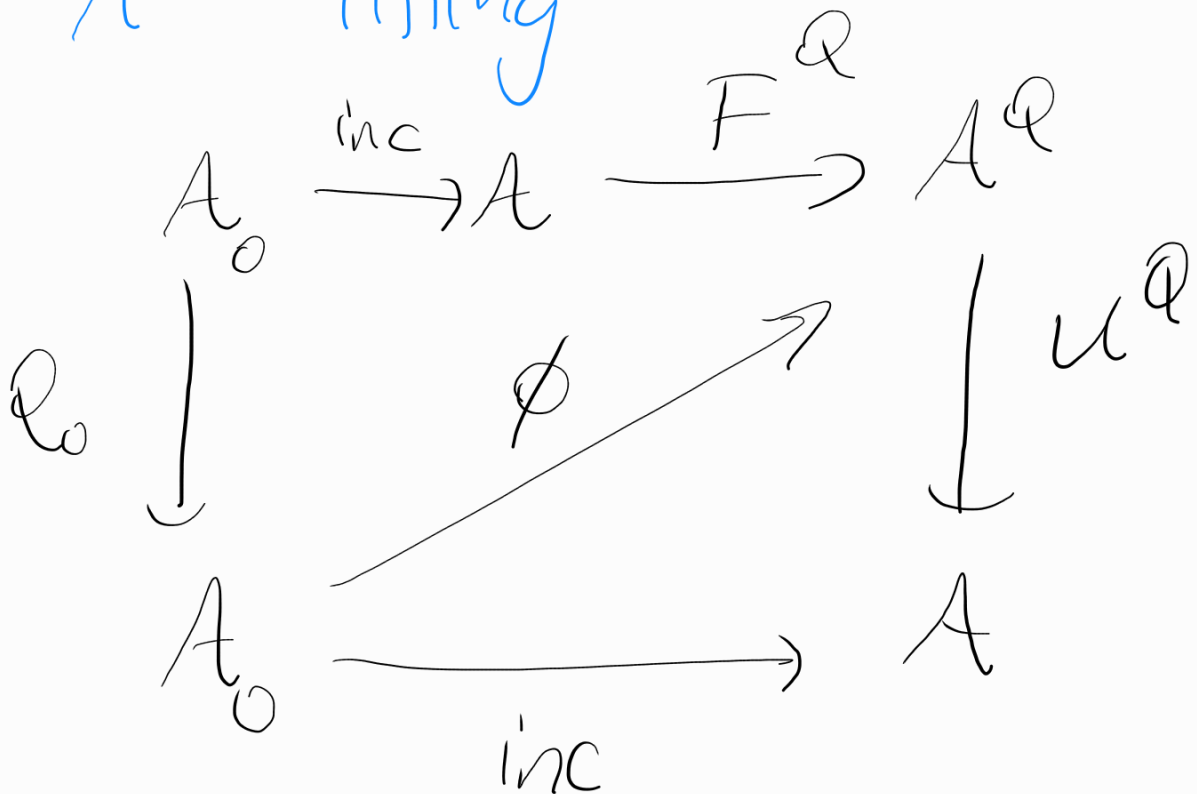


Part 3 Defining AKS on 2 categories.

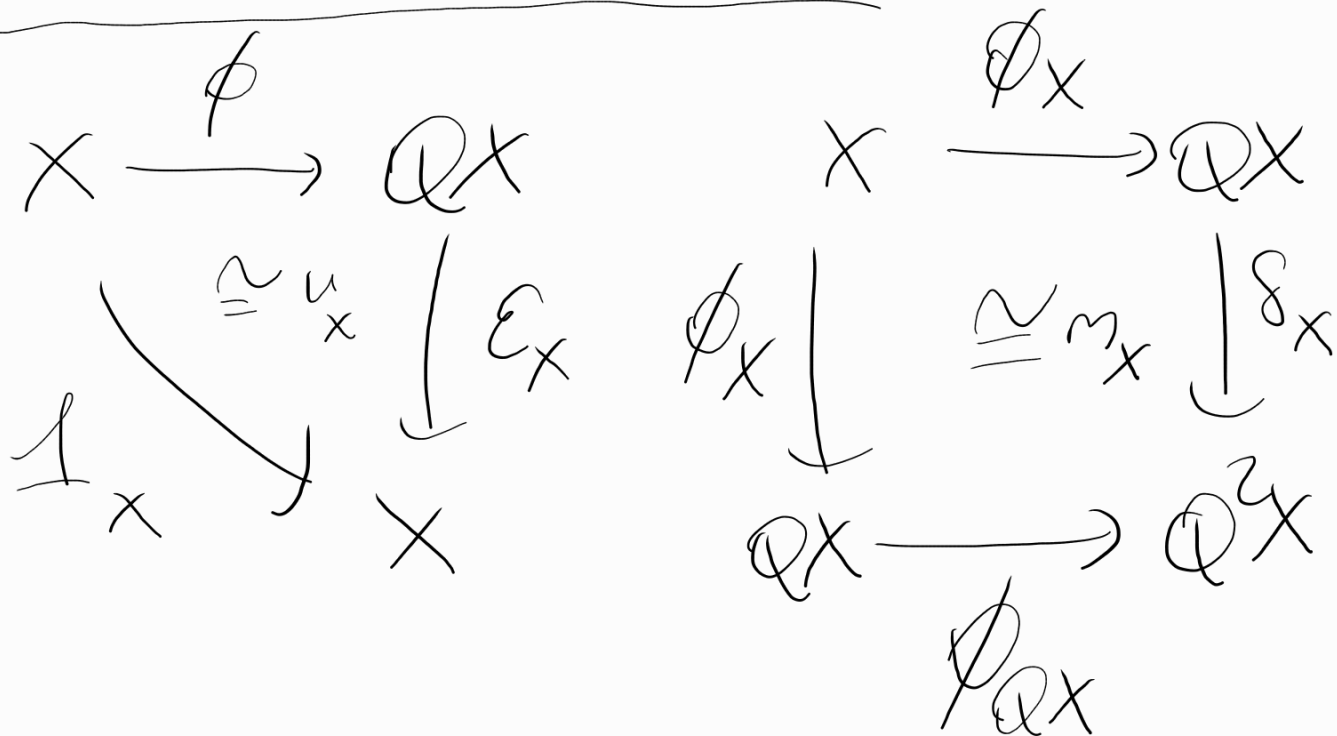
• A pseudocomonad

$$(A, Q, \epsilon, \delta, \eta, \theta, \rho)$$

• A lifting



In more detail?



such that  $\phi_{QX} = \mu_X$

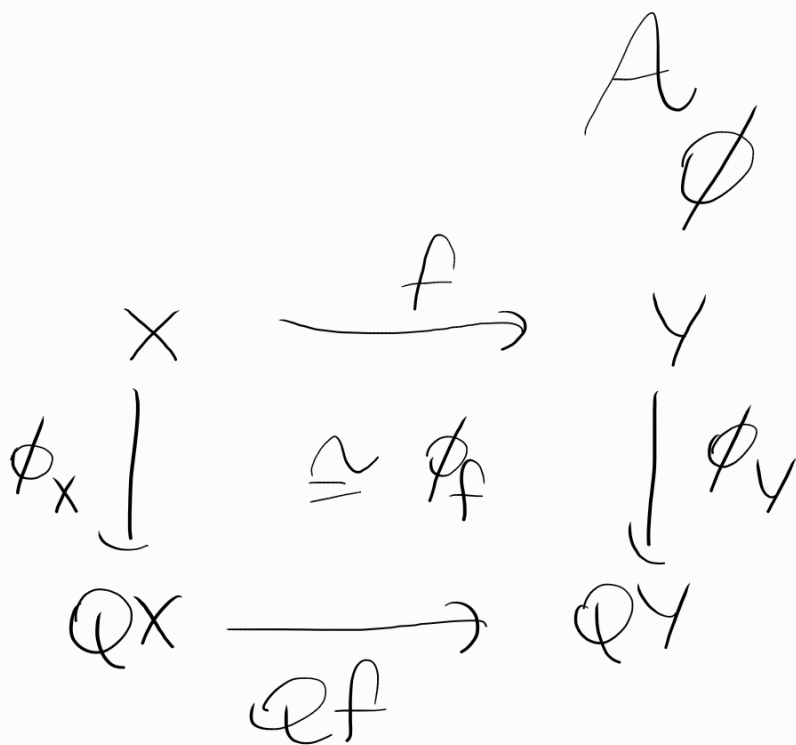
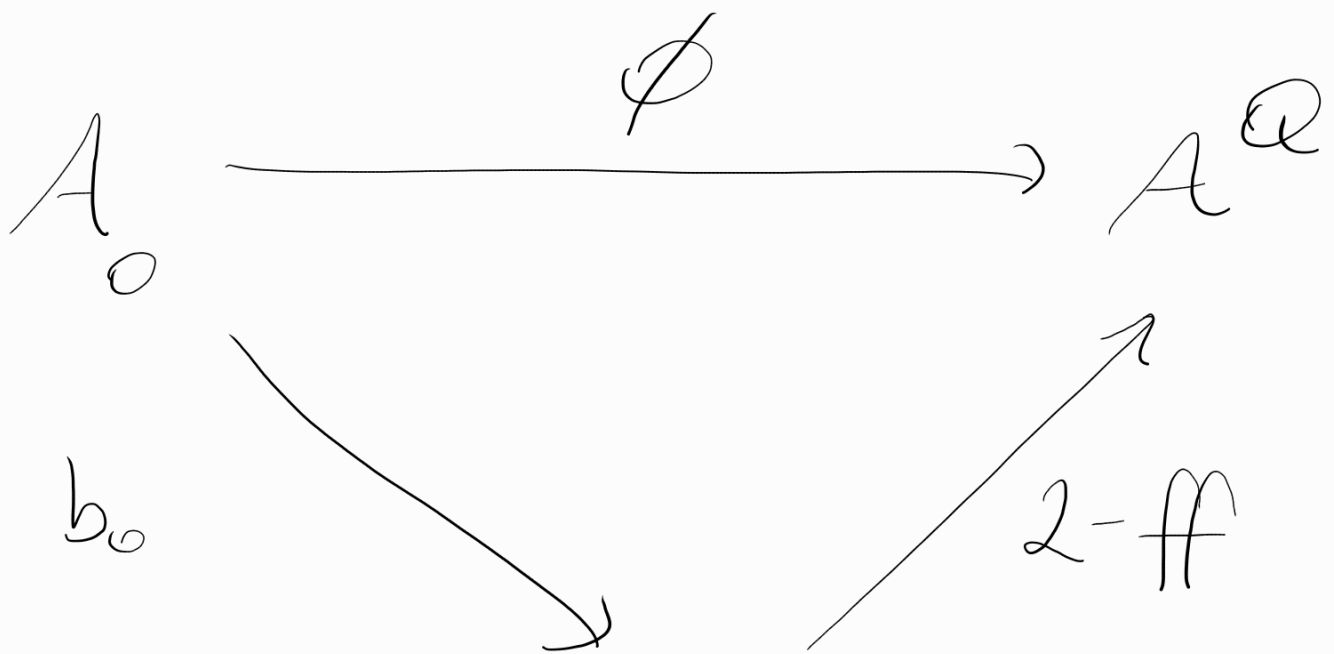
$$\mu_{QX} = \eta_X$$

$$m_{QX} = \eta_X$$

the pseudo coalgebra axioms hold.

Can form the 2-category

$A_\phi$  of "morphisms equipped with  
 thinking & thinkable 2-cells."



# Sidenotes

• Neither Kleisli bicats nor free ps-algebras are Kleisli objects in the enriched sense.

(Gambino, Lobbis 2022)

• Comparison from enriched Kleisli is a biequiv iff left adj is biess surj on objects.

(M. 2023)

Kleisli : free pseudalgebras.

$$A_{\emptyset} \begin{array}{c} \xleftarrow{+} \\ \xrightarrow{\quad} \end{array} A$$

pseudoadjunction

inducing  $(\eta, \epsilon, \delta, \tau, \gamma, \rho)$   
on  $A_{\emptyset}$

2-AKL  $\longrightarrow$  Pseudomonads

$$\begin{array}{ccc} A_{\emptyset} & \longrightarrow & B_{\psi} \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

pseudomonad  
on  $A_{\emptyset}$ .



Part 4 Analogous results for AKS  
on 2-categories

(M. 2024 Th<sup>m</sup> 6.1)

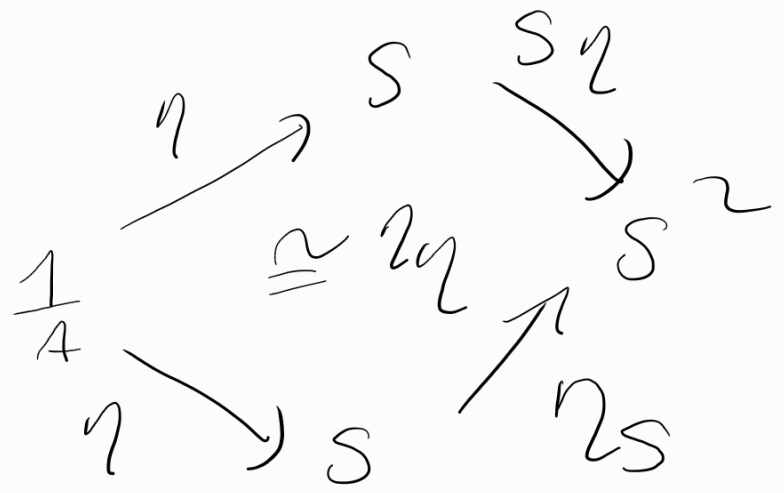
Have co-morphism of  
pseudomonads.

$$\begin{array}{ccc} A & \longrightarrow & (A_S)_S \\ \downarrow & & \downarrow \\ A_S & \xrightarrow{\quad \underline{1} \quad} & A_S \end{array}$$

which is a biequivalence

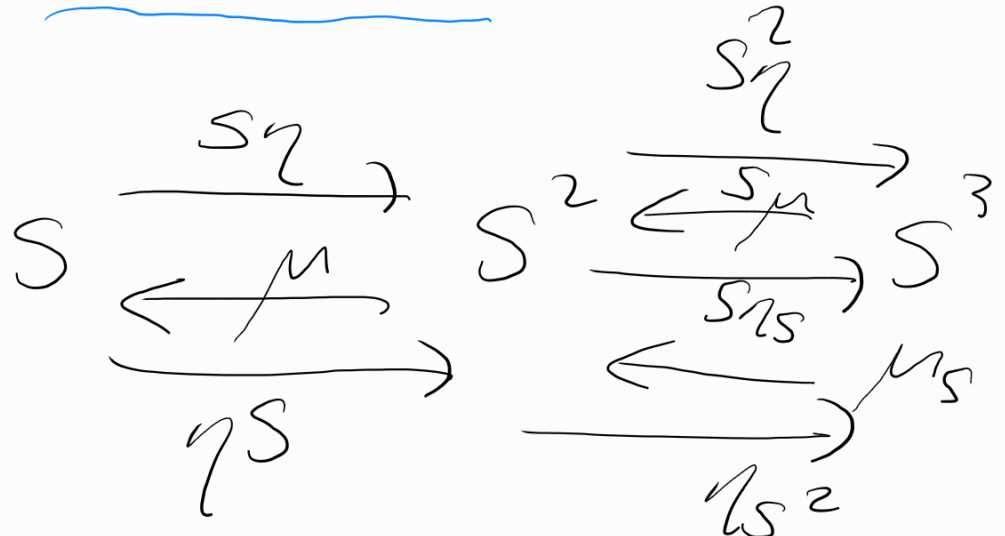
iff the following equivalent  
conditions hold

• The data



is an isobidescant cone

for



•  $A \rightarrow A_S$  faithful on 2-cells, full on thinkable 2-cells & on morphisms which admit a thinking.

•  $A \rightarrow (A_S)^S$  bi-fully faithful

•  $A \rightarrow (A^S)^S$  bi-fully faithful.

Th<sup>m</sup> 6.8 (M. 2024)

$$\begin{array}{ccc} A & \longrightarrow & (A_S)_{S_n} \\ \downarrow & & \downarrow \\ A_S & \xrightarrow{1} & A_S \end{array}$$

is the unit of

a reflection to

$$2\text{-AKS} \xrightarrow{\quad \text{I} \quad} \text{Pseudomonads}$$

NB. This is actually a Gray-adjunction.

# Future Work 1

- Fibrmann also studied monoidal versions
- I have recently shown how monoidal structures extend to Kleisli objects for pseudomonads
- To do: Study monoidal versions of AKS on 2-categories.

## Future Work 2

- I Konicoff & Lemay (Lemay 2023) used AKS to characterise when a (co)k-teisli' category supports structure of differentiation.
- I am currently developing the theory of differential bicategories
- Could seek a categorification of these results.