Abstract Kleisli structures on 2-categories

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Plan

1. Review & reformulate the definition of AKS on categories

2. Review & extend theory of AKS on categories

3. Define AKS on 2-categories

4. State analogous results for AKS on 2-categories.

Def^{n} (Führmann 1999)

An AKS consists of

$$\begin{tikzcd} A \arrow[r, bend left, "Q"] \arrow[r, bend right, "1_A"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & A \end{tikzcd}$$ "force"

$$X \xrightarrow[\phi_X]{} QX$$ "thunk"

such that $(A, Q, \varepsilon, \phi_Q)$ is a comonad & $\phi_X : X \longrightarrow QX$ is a coalgebra.

Part 1 Reformulating the definition of AKS on categories

$$\begin{tikzcd}
A_0 \arrow[r, "\mathrm{inc}"] \arrow[d, "Q_0"'] & A \arrow[r, "F^Q"] \arrow[d, phantom, "1"] & A^Q \arrow[d, "U^Q"] \\
A_0 \arrow[rr, "\mathrm{inc}"'] \arrow[urr, dashed, "\phi_-"] & \arrow[r, phantom, "2"] & A
\end{tikzcd}$$

1. ϕ_{QX} comultiplication for a comonad.

2. ϕ_X is a coalgebra.

- Note that $\phi_X : X \to QX$ not natural in X

- Morphisms $f : X \to Y$ satisfying

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi_X"'] & Y \arrow[d, "\phi_Y"] \\ QX \arrow[r, "Qf"'] & QY \end{tikzcd}$$

are called "thunkable."

- $$\begin{tikzcd} A \arrow[r, bend right, "T"'] & A \arrow[l, bend right, "Q"'] \end{tikzcd}$$ ϕ

  bijective on objects

- all examples are Kleisli categories of monads.

Part 2 Extending results on AKS on categories.

*

(1)

$$\begin{tikzcd} \mathrm{AKS} \arrow[rr, hook, bend right=10] \arrow[rr, phantom, "\perp"] & & \mathrm{Monads} \arrow[ll, bend right=30, "\mathrm{KL}"'] \end{tikzcd}$$

$$(A, Q, \varepsilon, \delta, \phi) \longmapsto A_\phi$$

(2) The essential image is characterised by any of the following equivalent conditions

- $X \xrightarrow{\eta_X} TX \overset{T\eta_X}{\underset{\eta_{TX}}{\rightrightarrows}} T^2X$ equalizer

- $B \longrightarrow B_T$ faithful & full on thunkable morphisms.

- $B \longrightarrow (B^T)^T$ fully faithful

- $B \longrightarrow (B_T)^T$ fully faithful.

* with strict morphisms of monads — Führmann (1999)

* with co-morphisms of monads & either possible 2-cell — (M. 2024) Prop 2.7.

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
A_S \arrow[r] & B_T
\end{tikzcd}
$$

co-morphism of monads

$$
\begin{tikzcd}
A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] \arrow[d] & B \arrow[d] \\
A_S \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B_T
\end{tikzcd}
$$

tight 2-cell

$$
\begin{tikzcd}
A \arrow[r, bend left] \arrow[r, bend right] \arrow[d] & B \arrow[d] \\
A_S \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B_T
\end{tikzcd}
$$

loose 2-cell

Part 3 Defining AKS on 2-categories.

- A pseudocomonad $(\mathcal{A}, Q, \varepsilon, \delta, \lambda, \alpha, \rho)$

- A lifting

$$\begin{tikzcd}
A_0 \arrow[r, "inc"] \arrow[d, "Q_0"'] & A \arrow[r, "F^Q"] & A^Q \arrow[d, "U^Q"] \\
A_0 \arrow[rru, "\phi"] \arrow[rr, "inc"'] & & A
\end{tikzcd}$$

In more detail:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, "1_X"'] & QX \arrow[d, "\varepsilon_X"] \arrow[dl, phantom, "\cong u_X"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\phi_X"] \arrow[d, "\phi_X"'] \arrow[dr, phantom, "\cong m_X"] & QX \arrow[d, "\delta_X"] \\ QX \arrow[r, "\phi_{QX}"'] & Q^2X \end{tikzcd}$$

such that
- $\phi_{QX} = \mu_X$
- $u_{QX} = \lambda_X$
- $m_{QX} = \alpha_X$

- the pseudocoalgebra axioms hold.

Can form the 2-category A_ϕ of "morphisms equipped with thunking & thunkable 2-cells":

$$\begin{tikzcd} A_0 \arrow[rr, "\phi"] \arrow[dr, "b_0"'] & & A^Q \\ & A_\phi \arrow[ur, "2\text{-ff}"'] & \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi_X"'] \arrow[dr, phantom, "\cong \phi_f"] & Y \arrow[d, "\phi_Y"] \\ QX \arrow[r, "Qf"'] & QY \end{tikzcd}$$

Sidenotes

- Neither Kleisli bicats nor free ps-algebras are Kleisli objects in the enriched sense.

(Gambino Lobbia 2022)

- Comparison from enriched kleisli is a biequiv iff left adj is biess surj on objects.

(M. 2023)

Kleisli : free pseudoalgebras.

$$\begin{tikzcd} A_\phi \arrow[r, shift right] \arrow[r, phantom, "\top"] & A \arrow[l, shift right] \end{tikzcd}$$

pseudoadjunction inducing $(Q, \varepsilon, \delta, \eta, \gamma, \rho)$ on A.

2-AKL $\longrightarrow$ Pseudomonads

$$\begin{tikzcd} A_\phi \arrow[r] \arrow[d] & B_\psi \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

pseudomonad on A_ϕ.

Part 4 Analogous results for AKS on 2-categories

(M. 2024 Th^m 6.1)

Have co-morphism of pseudomonads

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & (A_S)_{S_n} \arrow[d] \\
A_S \arrow[r, "1"'] & A_S
\end{tikzcd}$$

which is a biequivalence iff the following equivalent conditions hold

- The data

$$\begin{tikzcd}
 & S \arrow[dr, "S\eta"] & \\
1_A \arrow[ur, "\eta"] \arrow[dr, "\eta"'] \arrow[rr, phantom, "\cong\ 2\eta"] & & S^2 \\
 & S \arrow[ur, "\eta S"'] &
\end{tikzcd}$$

is an <u>isobidescent</u> cone

for

$$\begin{tikzcd}
S \arrow[r, shift left=4, "S\eta"] \arrow[r, shift right=4, "\eta S"'] & S^2 \arrow[l, "\mu"'] \arrow[r, shift left=6, "S^2\eta"] \arrow[r, "S\eta S"] \arrow[r, shift right=6, "\eta S^2"'] & S^3 \arrow[l, shift right=3, "S\mu"'] \arrow[l, shift left=3, "\mu S"]
\end{tikzcd}$$

- $A \longrightarrow A_S$ faithful on 2-cells, full on thunkable 2-cells & on morphisms which admit a thunking.
- $A \longrightarrow (A_S)^S$ bi-fully faithful
- $A \longrightarrow (A^S)^S$ bi-fully faithful.

Th^m 6.8 (M. 2024)

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & (A_S)_{Sn} \arrow[d, -|] \\
A_S \arrow[r, "1"] & A_S
\end{tikzcd}$$

is the unit of a reflection to

$$\begin{tikzcd}
\text{2-AKS} \arrow[r, ""{name=U, below}] & \text{Pseudomonads} \arrow[l, dashed, bend right=40, ""{name=D, above}] \arrow[from=D, to=U, phantom, "\perp"]
\end{tikzcd}$$

NB. This is actually a Gray-adjunction.

Future Work 1

- Führmann also studied monoidal versions
- I have recently shown how monoidal structures extend to Kleisli objects for pseudomonads
- To do: Study monoidal versions of AKS on 2-categories.

Future Work 2

- Ikonicoff & Lemay 2023 (Lemay 2023) used AKS to characterise when a (co)Kleisli category supports structure of differentiation.
- I am currently developing the theory of differential bicategories
- Could seek a categorification of these results.